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ON MAXIMUM AND MINIMUM VALUES OF THE MODULUS OF A POLYNOMIAL*

BY D. N. LEHMER

IN this note we consider the values of z of given modulus which give maximum or minimum values to the modulus of a given polynomial (rational integral function). Let the function be

$$f(z) = \sum_{i=0}^{n} p_i z^{n-i},$$

where each p_i is real. Put $z = \rho\ (\cos\theta + i \sin\theta)$. The corresponding value of the function is $x + i\,y$, where

$$\begin{aligned} x &= \sum_{i=0}^{n} p_i\, \rho^{n-i} \cos\,(n-i)\theta, \\ y &= \sum_{i=0}^{n} p_i\, \rho^{n-i} \sin\,(n-i)\theta. \end{aligned} \tag{1}$$

If the modulus of the point $x + i\,y$ is to be a minimum or a maximum for a given ρ, we must have

$$x\,\frac{dx}{d\theta} + y\,\frac{dy}{d\theta} = 0, \tag{2}$$

or, after a few trigonometric reductions,

$$\sum_{\substack{i=0\\ j=0}}^{n} p_i\, p_j\, \rho^{2n-i-j}\ (i-j)\sin\,(i-j)\theta = 0. \tag{3}$$

If the point z moves along the circumference of a circle of radius ρ with center at the origin, the point (x, y) will describe a curve parametrically given by equations (1). Equation (3) solved for θ gives the amplitudes of those points on this circle which correspond to the feet of normals from the origin on the curve described by (x, y).

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Suppose now ρ to vary. On each circle ρ there will be a certain number of points of the above description giving a locus of which (3) is the polar equation. This locus transforms into the locus of the feet of the normals from the origin upon the family of curves which corresponds to the system of circles ρ. From the equation (2) it is clear that this curve passes through the root points of $f(z)$.

THEOREM. *The lines joining the root points of $f(z)$ to the origin are tangent at those points to the curve whose polar equation is* (3), *or else form part of that curve.*

To prove this theorem we note that

$$\frac{\partial x}{\partial \rho}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \rho}\frac{\partial y}{\partial \theta} \equiv 0, \tag{4}$$

since, as is easily verified,

$$\frac{\partial x}{\partial \rho}\frac{\partial x}{\partial \theta} = - \frac{\partial y}{\partial \rho}\frac{\partial y}{\partial \theta} = -\sum p_i p_j\ (n-i)(n-j)\rho^{2n-i-j-1} \sin(2n-i-j)\,\theta.$$

Now from

$$x\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta} = 0$$

we get, differentiating with respect to ρ,

$$\frac{\partial x}{\partial \rho}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \rho}\frac{\partial y}{\partial \theta} + x\frac{\partial^2 x}{\partial \theta \partial \rho} + y\frac{\partial^2 y}{\partial \theta \partial \rho} = 0,$$

whence by (4) we have

$$x\frac{\partial^2 x}{\partial \rho \partial \theta} + y\frac{\partial^2 y}{\partial \rho \partial \theta} = 0,$$

which vanishes for $x = y = 0$. If therefore in equation (3) a value of θ be taken which will make the vector pass through a root point of $f(z)$, the equation will have a double root ρ equal to the modulus of that root. The vector ρ is therefore either a tangent to the curve or else forms part of the curve.

It is not difficult to prove the more general theorem that if p is a k-fold root of $f(z)$, the vector from the origin to p touches the curve (3) in $k + 1$ coincident points or else as before forms part of the curve.

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